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Series expansion analysis of directed site–bond percolation on the square and simple cubic lattices

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Abstract. Analysis of low-density series for site-bond percolation on the directed square (SQ) and simple cubic (SC) lattice (and related series for bond percolation on the honeycomb (H) and diamond (Di) lattices) is found to be consistent with

$p_c(\text{SQ, site-bond}) = p_c(\text{H, bond}) = 0.8228 \pm 0.0002$

$p_c(\text{SC, site-bond}) = p_c(\text{Di, bond}) = 0.637 \pm 0.002$

and previous estimates of γ, $\nu_\|$ and $\nu_\perp$. Analysis of the square lattice series supports the validity of the scaling relation

$\gamma_0 = \gamma - (d-1)\nu_\perp$

for the two-dimensional lattices.

Site percolation on the honeycomb and diamond lattices is also considered.

1. Introduction

The statistical properties of the directed percolation problem, in which bonds (and/or sites) of a lattice are occupied with probability p and fluid flow is restricted so that it always has a positive component along some chosen axis (the preferred direction), may be determined from a knowledge of the pair-connectedness $C_i(p)$, the probability that site i is connected to the origin. The moments of the pair-connectedness are given by

$$\mu_{l,m}(p) = \sum_i x_i^l t_i^m C_i(p), \tag{1}$$

where x_i and t_i are the components of the position vector of site i perpendicular to and parallel to the preferred direction of fluid flow respectively. The moment $\mu_{00}(p)$ is the mean size $S(p)$ of the cluster connected to the origin. For p sufficiently close to its critical value p_c the moments are assumed to have the asymptotic form

$$\mu_{l,m}(p) \sim |p_c - p|^{-\gamma - l\nu_\perp - m\nu_\|} \tag{2}$$

(Cardy and Sugar 1980, Kinzel and Yeomans 1981). Moreover the scaling form for the pair-connectedness proposed by Cardy and Sugar (1980) leads to the scaling prediction

$$\gamma_0 = \gamma - (d-1)\nu_\perp \tag{3}$$

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where γ_0 is the critical exponent of the diagonal mean size

$$S_0(p)=\sum_{i:x_i=0} C_i(p) \tag{4}$$

and d is the lattice dimension.

In this paper we extend our previous work on two- and three-dimensional directed percolation (De'Bell and Essam 1983a, b, hereafter referred to as I and II) to site-bond percolation on the directed square and simple cubic lattices (§ 2). Previous series expansion work on site-bond percolation has been for undirected lattices (Agrawal *et al* 1979, Brown *et al* 1975). In site-bond percolation both sites and bonds are independently present with probability p, which means that the same configurational data gives rise to series which are twice as long as those for the corresponding site and bond problems. We have used these series to test relation (3) by comparing estimates of $\nu_\perp$ and ν_0, defined by

$$\nu_0=(\gamma-\gamma_0)/(d-1), \tag{5}$$

which should be equal if (3) is true. The results of previous tests (given in I and II) showed significant differences between ν_0 and $\nu_\perp$ which were relatively small in two dimensions but quite pronounced in three dimensions. The latter was attributed to the special nature of the $S_0(p)$ series which has a length which is effectively much shorter than that of the other moments. This effect is much worse in three dimensions where the number of available coefficients is in any case rather small. We shall find that the data for directed site-bond percolation strongly supports equation (3) in two dimensions but that the discrepancy remains in three dimensions.

In § 3 we consider series expansions for directed bond and site percolation on the honeycomb and diamond lattices. The bond percolation series may be derived from the site-bond series above whereas the site problem series are determined by previously published (I and II) site problem series for the square and simple cubic lattices (Essam and De'Bell 1982).

Our results are summarised in table 1 and are based on Padé approximant analysis of the series expansions tabulated in the appendix. The coefficients in these expansions were obtained by the methods described in I.

Table 1. Summary of critical probabilities and exponents for site-bond percolation. The coefficients of Δp_c are obtained from the tangent to the pole-residue curve at the estimated value p_c and measure the sensitivity of the exponent values to changes in this estimate.

	Square lattice	Simple cubic lattice
p_c	$0.82281+0.01\Delta\gamma\pm0.00002$	0.637 ± 0.002
γ	2.269 (assumed)	$1.575+44\Delta p_c\pm0.003$
$\nu_\parallel$	$1.731+70\Delta p_c\pm0.004$	$1.260+19\Delta p_c\pm0.003$
$\nu_\perp$	$1.100+51\Delta p_c\pm0.005$	$0.728+14\Delta p_c\pm0.002$
ν_0	$1.097+36\Delta p_c\pm0.001$	$0.638+12\Delta p_c\pm0.004$

2. Analysis of directed site–bond percolation series

As usual (Gaunt and Guttman 1974) we form a selection of Padé approximants to the logarithmic derivative (Dlog) of various series in order to estimate p_c and the exponents γ, $\nu_\parallel$, $\nu_\perp$ and ν_0.

The pole-residue data for the square lattice Dlog $S(p)$ series is given in table 2. Our estimate of p_c (table 1) derived from this data is biased slightly upwards to give $\gamma = 2.269$ and is in agreement with that obtained by Kinzel and Yeomans (1981) using finite size scaling techniques. This value of γ was obtained from the square lattice bond problem series which showed the best convergence of the series examined in I. The term $0.01\Delta\gamma$ (table 1) shows the sensitivity of p_c to changes in the assumed value of γ and the error in the value of p_c quoted in the abstract is obtained by taking $|\Delta\gamma| \leq 0.02$ as in I. Similar pole-residue data from $\mu_{2,0}/S$ $(\sim(p_c-p)^{-2\nu_\perp})$ and S/S_0 $(\sim(p_c-p)^{\gamma_0-\gamma})$ which has been scaled to give estimates of $\nu_\perp$ and ν_0 respectively are shown in figure 1. The points lie on two distinct curves which cross in the vicinity of the estimated p_c in excellent agreement with the scaling prediction $\nu_0 = \nu_\perp$ which is equivalent to (3) by definition of ν_0. This relation has also recently been confirmed by Monte Carlo data for directed site percolation on the square lattice (De'Bell *et al* 1984).

Table 2. Poles and residues of the Dlog Padé approximants from the mean size series for site-bond percolation on the square lattice.

	$(N/N-2)$		$(N/N-1)$		(N/N)		$(N/N+1)$		$(N/N+2)$	
N	p_c	γ	p_c	γ	p_c	γ	p_c	γ	p_c	γ
21	—	—	0.8225	2.241	0.8225	2.239	0.8228	2.272	0.8228	2.263
22	0.8225	2.242	0.8224	2.234	0.8228	2.263	0.8229[D]	2.275	0.8227	2.255
23	0.8213[D]	2.248	0.8226	2.246	0.8227	2.254	0.8227	2.262	—	—
24	0.8228	2.267	0.8223[D]	2.228						

[D] These approximants have an interfering defect and should be ignored when estimating p_c or γ.

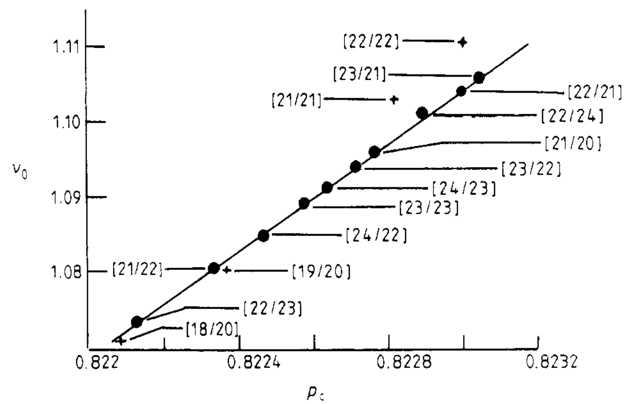

Figure 1. Estimates of ν_0 and $\nu_\perp$ for site-bond percolation on the square lattice: (●) pole-residue plot for Dlog(S/S_0); (+) the four points which are closest to p_c from the pole-residue plot for $\frac{1}{2}[\mathrm{Dlog}(\mu_{2,0}/S)]$.

In the analysis of three-dimensional bond and site percolation presented in II no particular series was chosen as giving the best estimate of γ and the value of p_c for site-bond percolation on the simple cubic lattice given in table 1 is unbiased. The result quoted is based on the data in table 3 which includes pole-residue pairs from

Table 3. Poles and residues of the Dlog Padé approximants for site-bond percolation on the simple cubic lattice.

(a) Mean size S.

	$(N/N-2)$		$(N/N-1)$		(N/N)		$(N/N+1)$		$(N/N+2)$	
N	p_c	γ	p_c	γ	p_c	γ	p_c	γ	p_c	γ
9	—	—	0.6398^D	1.669	0.6383	1.618	0.6364^D	1.542	0.6208	0.777
10	0.6399^D	1.670	0.6416^D	1.711	0.6096	3.894	0.6339	1.437	0.6383	1.624
11	0.6379	1.613	0.6392	1.653	0.6375	1.594	0.6369	1.570	—	—
12	0.6385	1.630	0.6363	1.538						

(b) $\mu_{0,2}/S$.

	$(N/N-2)$		$(N/N-1)$		(N/N)		$(N/N+1)$		$(N/N+2)$	
N	p_c	$2\nu_\parallel$	p_c	$2\nu_\parallel$	p_c	$2\nu_\parallel$	p_c	$2\nu_\parallel$	p_c	$2\nu_\parallel$
7	—	—	—	—	0.6375	2.539	0.6373	2.532	0.6376	2.545
8	—	—	0.6380	2.555	0.6377	2.544	0.6374	2.536	0.6372	2.529
9	0.6367	2.507	0.6382	2.562	0.6372	2.529	0.6377^D	2.544	0.6420^D	2.54
10	0.6423^D	2.676	0.6364	2.502	0.6413^D	2.55	0.6367	2.519	—	—
11	0.6353	2.458	0.6342	2.405						

^{D} Interfering defect.

both the mean size series and the series for $\mu_{0,2}/S$. There appear to be no earlier estimates of the critical probability for this problem. The estimates of $\nu_\perp$ and ν_0 (table 1) are in good agreement with those reported in II and the apparent inconsistency with (3) is discussed therein. It can be seen that the error in the three-dimensional exponent estimates is dominated by the relatively large uncertainty in p_c which is ten times greater than in two dimensions.

3. The honeycomb and diamond lattices

It has been shown (Essam and De'Bell 1982) that the moments for site-bond percolation on the directed square and cubic lattice determine the moments $\hat{\mu}_{l,m}$ for bond percolation on the directed honeycomb and diamond lattices using the relation

$$\hat{\mu}_{l,m}(p) = \sum_i [d(d+1)x_i^2]^{l/2} d^{m/2} C_i(p)\{[(d+1)t_i]^m + p[(d+1)t_i + d^{1/2}]^m\} \qquad (6)$$

with $d=2$ and 3. The result also holds for a general directed 'hyperdiamond' lattice defined as follows. Consider a d-dimensional cubic lattice and contract it uniformly in the $d-1$ dimensions perpendicular to the $(1, 1, \ldots)$ axis until each site, together with its d nearest neighbours with non-negative coordinates, forms a hypertetrahedron. If an extra site is placed at the centre of each hypertetrahedron so formed, the resulting structure is one in which each site has $d+1$ nearest neighbours. Finally the nearest-neighbour bonds are all directed in the positive sense relative to the $(1, 1, \ldots)$ axis. The moments $\hat{\mu}_{l,m}$ on the left of (6) are calculated relative to an origin on the original cubic lattice. The pair-connectedness $C_i(p)$ in (6) is for site-bond percolation on the hypercubic lattice and the sum is over sites on this lattice.

In the case $m=0$, equation (6) leads to the simple relation

$$\hat{\mu}_{l,0}(p)=[d(d+1)]^{1/2}(1+p)\mu_{l,0}(p) \tag{7}$$

which establishes equality of the critical probabilities for bond percolation on the hyperdiamond and site-bond percolation on the hypercubic lattices. The value of p_c for bond percolation on the directed honeycomb lattice which may consequently be read from table 1 represents a slight upward revision of the value $p_c=0.8226\pm 0.0002$ obtained by Blease (1977) using the first forty-three terms of the mean size series. The increase results from the previously mentioned imposition of the value $\gamma=2.269$. Since $\nu_\perp$ and ν_0 are normally determined from moment ratios the Padé tables corresponding to these exponents for bond percolation on the hyperdiamond lattices would (using (7)) be identical to those of the corresponding hypercubic problems.

The expression for the second moment $\hat{\mu}_{0,2}$ which is normally used to determine $\nu_\parallel$ involves three of the cubic moments,

$$\hat{\mu}_{0,2}(p)=d(d+1)^2(1+p)\mu_{0,2}(p)+dp[2(d+1)d^{1/2}\mu_{0,1}(p)+d\mu_{0,0}(p)] \tag{8}$$

and the resulting series for the honeycomb and diamond lattices are given in the appendix. The moments $\mu_{0,0}$ and $\mu_{0,1}$ are less strongly divergent at p_c than $\mu_{0,2}$ but nevertheless a Padé analysis of $\hat{\mu}_{0,2}/\hat{\mu}_{0,0}$ rather than $\mu_{0,2}/\mu_{0,0}$ gives a different set of data from which to estimate $\nu_\parallel$. It is found that the pole-residue pairs for the honeycomb and diamond bond problems lie on the same curves as for the corresponding site-bond problems so that the $\nu_\parallel$ estimates in table 1 also apply to these problems.

Equation (6) also determines the moments for site percolation on the hyperdiamond lattice (Essam and De'Bell 1982) but now $C_i(p)=C_i^*(p^2)$ where $C_i^*(p)$ is the pair-connectedness for site percolation on the hypercubic lattice. The required site percolation moment series on the square and simple cubic lattices are given in I and II respectively with the exception of the first moment series which are given in the appendix. The resulting series for $\mu_{0,2}(p)$ on the honeycomb and diamond lattices are also listed in the appendix. Conversion of our previous p_c results for site percolation on the square and simple cubic lattices gives

$$\begin{aligned} p_c(\text{H, site}) &= p_c^{1/2}(\text{SQ, site}) \\ &= 0.8399 \bullet 0.0001 \end{aligned}$$

and

$$\begin{aligned} p_c(\text{Di, site}) &= p_c^{1/2}(\text{SC, site}) \\ &= 0.659\pm 0.003. \end{aligned}$$

Again (7) implies that the estimates of $\nu_\perp$ and ν_0 based on $\mu_{2,0}/S$ and S/S_0 will be the same as those for the square and simple cubic site problems given in I and II. Analysis of the $\mu_{0,2}(p)$ series gives estimates of $\nu_\parallel$ similar to those in I and II.

4. Concluding remarks

It is generally believed that site and bond percolation are in the same universality class and that site-bond percolation will also belong to this class. This has been demonstrated for undirected percolation by series (Agrawal *et al*) and position-space renormalisation group (Nakanishi and Reynolds 1979) methods. Our results for directed percolation are clearly consistent with this universality. It was hoped that the exponents for the above class would be more accurately determined by the much longer series for site-bond percolation. This turned out not to be the case although the evidence for the validity of the scaling relation (3) in two dimensions was much stronger than that found in I. The inconsistency in three dimensions remains but we still believe this to be due to the special nature of the S_0 series referred to in II.

Appendix. Coefficients of p^n in the low-density series

Table A1. The site-bond problem on the directed square lattice.

n	S	S_0	$\sqrt{2}\mu_{0,1}$	$\frac{1}{2}\mu_{2,0}$	$\mu_{0,2}$
0	1	1	0	0	0
1	0	0	0	0	0
2	2	0	2	1	1
3	0	0	0	0	0
4	4	2	8	4	8
5	0	0	0	0	0
6	8	0	24	12	36
7	−1	−1	−2	0	−2
8	16	6	64	32	128
9	−4	0	−12	−2	−18
10	32	0	160	80	400
11	−14	−6	−54	−13	−105
12	66	20	390	193	1 161
13	−40	−4	−192	−56	−464
14	137	5	932	456	3 208
15	−109	−33	−620	−200	−1780
16	294	74	2 230	1 071	8 631
17	−280	−26	−1 844	−638	−6 138
18	640	40	5 332	2 506	22 802
19	−706	−168	−5 250	−1 893	−19 793
20	1 429	301	12 864	5 902	59 798
21	−1 737	−175	−14 382	−5356	−60 502
22	3 234	286	31 208	13 974	156 078
23	−4 246	−852	−38 544	−14 626	−178 318
24	7 448	1 356	76 408	33 408	407 376
25	−10 286	−1 074	−101 342	−38 997	−510 249
26	17 334	1 808	188 192	80 492	1 063 380
27	−24 872	−4 370	−263 410	−102 103	−1 429 587
28	40 755	6 475	466 420	195 606	2 779 686
29	−59 964	−6 458	−677 906	−263 986	−3 936 828
30	96 531	10 989	1 160 942	478 491	7 271 631
31	−144 713	−22 793	−1 734 724	−676 142	−10 706 946
32	230 116	33 040	2 900 508	1 177 576	19 037 632
33	−349 177	−38 399	−4 416 598	−1 720 494	−28 820 056
34	551 227	64 909	7 264 796	2 911 098	49 846 302

Table A1. (*continued*)

n	S	S_0	$\sqrt{2}\mu_{0,1}$	$\frac{1}{2}\mu_{2,0}$	$\mu_{0,2}$
35	−844 026	−121 918	−11 213 290	−4 357 317	−76 989 877
36	1 325 875	175 585	18 232 196	7 223 650	130 492 310
37	−2 042 320	−227 558	−28 398 442	−10 999 602	−204 370 932
38	3 200 362	380 442	45 821 496	17 974 782	341 420 726
39	−4 952 069	−665 233	−71 832 388	−27 706 907	−539 954 737
40	7 747 369	968 457	115 278 538	44 826 469	892 602 561
41	−12 025 029	−1 345 581	−181 485 294	−69 687 927	−1 420 862 891
42	18 803 789	2 216 501	290 247 646	111 975 445	2 331 392 965
43	−29 257 829	−3 709 379	−458 312 338	−175 126 984	−3 727 544 850
44	45 741 700	5 463 010	731 205 898	280 085 839	6 083 098 175
45	−71 299 218	−7 951 882	−1 156 868 020	−439 867 603	−9 753 207 325
46	111 502 853	12 944 433	1 842 981 826	701 306 807	15 855 341 443
47	−174 061 514	−21 027 246	−2 919 904 730	−1 104 601 543	−25 466 801 915
48	272 304 224	31 426 920	4 646 879 872	1 757 501 424	41 282 335 480

Table A2. The site-bond problem on the directed simple cubic lattice.

n	S	S_0	$\sqrt{3}\mu_{0,1}$	$\frac{1}{2}\mu_{2,0}$	$\mu_{0,2}$
0	1	1	0	0	0
1	0	0	0	0	0
2	3	0	3	1	1
3	0	0	0	0	0
4	9	0	18	6	12
5	0	0	0	0	0
6	27	6	81	27	81
7	−3	0	−6	−1	−4
8	81	0	324	108	432
9	−18	−6	−54	−12	−54
10	243	0	1 215	405	2 025
11	−96	−9	−369	−87	−477
12	741	96	4 410	1 464	8 784
13	−414	0	−1 980	−498	−3 180
14	2 280	12	15 669	5 169	36 189
15	−1 716	−186	−9 693	−2 508	−18 453
16	7 160	2	55 287	18 055	144 373
17	−6 627	−297	−43 401	−11 548	−95 857
18	22 827	1 992	194 415	62 705	563 477
19	−25 219	−478	−186 312	−50 265	−465 816
20	74 220	1 098	686 070	218 006	2 172 044
21	−93 111	−6 378	−766 329	−209 396	−2 139 209
22	245 019	2 817	2 430 999	760 135	8 307 193
23	−341 853	−12 366	−3 083 418	−847 429	−9 466 198
24	822 708	52 965	8 677 386	2 667 024	31 678 578

Table A3. Low-density expansions for additional longitudinal moments.

	$\mu^{*}_{0,1}(p^2)$		$\hat{\mu}_{0,2}(p)$			
	Square	Cubic	Honeycomb		Diamond	
n	site	site	Bond	Site	Bond	Site
1	2	3	4	4	9	9
2	8	18	18	18	48	48
3	22	75	50	50	147	147
4	52	270	144	144	576	576
5	112	882	256	256	1 089	1 089
6	228	2 736	648	612	3 888	3 696
7	442	8 085	932	904	5 883	5 712
8	832	23 334	2 240	1 980	20 373	18 144
9	1 516	65 184	2 812	2 652	26 649	25 191
10	2 720	180 186	6 716	5 472	93 150	76 032
11	4 754	485 202	7 358	6 896	105 651	98 631
12	8 264		18 304	13 680	389 016	288 864
13	14 000		17 490	16 548	381 501	358 146
14	23 824		46 928	31 734	1 533 186	1 019 088
15	39 318		37 436	37 250	1 247 904	1 222 020
16	66 052		115 442	69 804	5 796 084	3 419 808
17	106 282		72 810	80 128	3 720 096	4 001 910
18	177 884		276 704	146 718	21 344 493	10 976 832
19	277 936		120 706	165 442	9 559 131	12 594 546
20	469 384		654 266	298 548	77 200 485	34 210 560
21	703 924		147 412	332 028	18 709 740	38 666 028
22	1 225 052		1 540 836	588 402	276 833 337	103 373 568
23			−12 888	646 738	4 916 907	115 330 185
24			3 643 532	1 136 016	989 115 531	
25			−905 026	1 237 204		
26			8 699 110	2 138 400		
27			−4 264 086	2 309 436		
28			21 041 374	3 975 984		
29			−15 068 496	4 266 640		
30			51 651 726	7 225 380		
31			−47 518 242	7 704 188		
32			128 556 808	13 067 388		
33			−140 357 072	13 871 140		
34			324 076 544	23 070 204		
35			−399 201 890	24 361 312		
36			825 108 210	40 898 736		
37			−1 105 725 344	43 059 248		
38			2 117 945 708	70 362 702		
39			−3 010 952 798	73 732 250		
40			5 465 863 900	123 163 056		
41			−8 094 354 008	128 856 528		
42			14 163 617 688	206 444 502		
43			−21 572 547 022	214 962 978		
44			36 783 180 478	359 875 872		
45			−57 104 527 124	374 723 540		
46			95 670 801 012			
47			−150 444 495 172			
48			248 944 501 354			

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